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THE MOTION OF A RIGID BODY IN A FLUID UNDER THE ACTION OF CENTRAL NEWTONIAN ATTRACTIVE FORCES†

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The structure of the forces and moments which arise when a rigid body moves in an unbounded volume of an ideal incompressible fluid under the action of central Newtonian attractive forces is discussed, the equations of motion are written in explicit form, their first integrals are indicated and the properties of these integrals are investigated. The problem of possible simplifications in the formulation of the problem is investigated, based on an analogy with the classical "satellite approximation" and which hold when the translational velocity of the body can be regarded as being independent of its rotational motion. The question of the existence and stability of steady motions, in this case relative equilibria, is investigated in the satellite approximation. © 2001 Elsevier Science Ltd. All rights reserved.

The motion of a rigid body in a central Newtonian force field is the classical subject of investigations in theoretical mechanics (see, for example, [1, 2]). Numerous investigations in this area have enabled a fairly complete representation of the properties of the motion of such a system to be established. Nevertheless, the properties of the motion of the same system in a space filled with an ideal incompressible fluid at rest at infinity have considerable differences. The reason for this is the interaction of the fluid and body during the motion, which manifests itself in the fact that the vectors of translational and angular velocity of the body depend on one another.

1. THE LAGRANGIAN OF THE STRUCTURE AND THE FIRST INTEGRALS OF THE EQUATIONS OF MOTION

Consider the motion of a rigid body $\mathcal{S}$ in an ideal incompressible fluid, which fills the whole of space and is at rest at infinity. We will assume that the system moves under the action of central Newtonian attractive forces with centre at the point N, fixed in absolute space. Suppose $NX_\alpha X_\beta X_\gamma$ is the absolute system of coordinates, C is a point fixed in the body and $Cx_1x_2x_3$ is a moving reference frame connected with the body. Suppose also that

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \quad \boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3), \quad \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$$

are the unit vectors of the inertial system of coordinates $NX_\alpha X_\beta X_\gamma$ and $\mathbf{r} = (r_1, r_2, r_3)$ is the vector $\overrightarrow{NC}$. Suppose $\omega = (\omega_1, \omega_2, \omega_3)$ is the absolute angular velocity of the body and $\mathbf{v} = (v_1, v_2, v_3)$ is the absolute velocity of the point C. Here and henceforth, unless otherwise stated, all the vectors and tensors are specified by their coordinates in the moving reference frame. The equations which express the theorems on the change in momentum and on the change in the angular momentum can be written in the form of the Lagrange–Euler–Poincaré equations

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}} = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial \mathbf{v}} \times \mathbf{v} + \frac{\partial L}{\partial \mathbf{r}} \times \mathbf{r} \\ &\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{v}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial \mathbf{r}} \end{aligned} \tag{1.1}$$

These equations must be supplemented with the kinematic equations

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$$\dot{\mathbf{r}} = \mathbf{v} + \mathbf{r} \times \boldsymbol{\omega} \tag{1.2}$$

which express the variation in the vector $\mathbf{r}$ with respect to the moving reference frame.

Lagrange's function is written, as usual, as the difference between the kinetic and potential energies

$$L(\boldsymbol{\omega}, \mathbf{v}, \mathbf{r}) = T(\boldsymbol{\omega}, \mathbf{v}) - U(\mathbf{r}) \tag{1.3}$$

After Eqs (1.1) and (1.2) have been integrated, we can obtain the change in the orientation of the body by integrating Poisson's equations

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega} \tag{1.4}$$

Equations (1.1) and (1.2) with Lagrange's function, which does not depend explicitly on the time, allow of a (Penlevé–Jacobi) first energy integral

$$\mathcal{J}_0 = \left(\frac{\partial L}{\partial \boldsymbol{\omega}}, \boldsymbol{\omega}\right) + \left(\frac{\partial L}{\partial \mathbf{v}}, \mathbf{v}\right) - L = h \tag{1.5}$$

Moreover, by virtue of Eqs (1.1) and (1.2) we have the relation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}} + \mathbf{r} \times \frac{\partial L}{\partial \mathbf{v}}\right) = \left(\frac{\partial L}{\partial \boldsymbol{\omega}} + \mathbf{r} \times \frac{\partial L}{\partial \mathbf{v}}\right) \times \boldsymbol{\omega} \tag{1.6}$$

which expresses the law of variation of the angular momentum vector of the system with respect to the moving reference frame. In other words, the total angular momentum remains unchanged in absolute space, so that its projection onto any direction, fixed in absolute space, also remains unchanged. Each of these projections is the first integral of the equations of motion. We will choose from them three independent ones

$$\mathcal{J}_i = \left(\frac{\partial L}{\partial \boldsymbol{\omega}} + \mathbf{r} \times \frac{\partial L}{\partial \mathbf{v}}, \ \mathbf{i}\right), \quad \mathbf{i} \in \{\boldsymbol{\alpha}, \ \boldsymbol{\beta}, \ \boldsymbol{\gamma}\} \tag{1.7}$$

which express the projection of the total angular momentum vector onto the axis of the absolute system of coordinates. For each motion, the axes of this reference frame can be chosen in such a way that at the initial instant the total angular momentum vector is directed, say, along the β axis.

We obtain

$$\mathcal{J}_\alpha = 0, \ \ \mathcal{J}_\beta = p_\psi, \ \ \mathcal{J}_\gamma = 0 \tag{1.8}$$

The integrals $\mathcal{J}_\alpha$, $\mathcal{J}_\beta$, $\mathcal{J}_\gamma$ and the six geometrical integrals of Poisson's equations, which express the orthonormality of the reference frame $NX_\alpha X_\beta X_\gamma$ and which have the form

$$\begin{aligned} &\mathcal{J}_{ii} = (\mathbf{i}, \mathbf{i}) - 1 = 0, \ \ \mathbf{i} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}, \\ &\mathcal{J}_{ij} = (\mathbf{i}, \mathbf{j}) = 0, \ \ \mathbf{i}, \mathbf{j} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}, \ \ \mathbf{i} \neq \mathbf{j} \end{aligned} \tag{1.9}$$

enable us to reduce the order of the system of equations of motion and to reduce the problem to the integration of Lagrange's equations with five degrees of freedom. To integrate it, in addition to the energy integral we also need to know four independent integrals which commute with one another.

The equations of motion also allow of the integral

$$\mathcal{J}_1 = \left(\frac{\partial L}{\partial \boldsymbol{\omega}} + \mathbf{r} \times \frac{\partial L}{\partial \mathbf{v}}, \ \ \frac{\partial L}{\partial \boldsymbol{\omega}} + \mathbf{r} \times \frac{\partial L}{\partial \mathbf{v}}\right) \tag{1.10}$$

which expresses the square of the total angular momentum vector or, which is the same thing, the sum of the squares of the integrals $\mathcal{J}_\alpha$, $\mathcal{J}_\beta$, and $\mathcal{J}_\gamma$. The value of this integral, the existence of which can be seen from system (1.6), unlike the values of the integrals (1.7), does not depend on the choice of the absolute system of coordinates.

2. THE KINETIC ENERGY

We will investigate the structure of Lagrange's function (1.3) in more detail. As is well known, the motion of a rigid body is described by the equations

$$\frac{d}{dt}\frac{\partial T_c}{\partial \boldsymbol{\omega}} = \frac{\partial T_c}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial T_c}{\partial \boldsymbol{\omega}} \times \mathbf{v} + \mathbf{M} \tag{2.1}$$

$$\frac{d}{dt}\frac{\partial T_c}{\partial \mathbf{v}} = \frac{\partial T_c}{\partial \mathbf{v}} \times \boldsymbol{\omega} + \mathbf{F} \tag{2.2}$$

where T_c is the kinetic energy of the body, and $\mathbf{F}$ and $\mathbf{M}$ are the force and the moment of the forces acting on the body. In the case considered

$$T_c = \frac{1}{2}((\mathbf{I}_{\mathcal{G}}\boldsymbol{\omega}, \boldsymbol{\omega}) + 2(\mathbf{B}_{\mathcal{G}}\boldsymbol{\omega}, \mathbf{v}) + M_{\mathcal{G}}(\mathbf{v}, \mathbf{v}))$$

where $\mathbf{I}_{\mathcal{G}}$ is the inertia tensor of the body with respect to the point C. If $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ is the vector $\overrightarrow{CG}$, where G is the centre of mass of the body, $\mathbf{E}$ is the identity 3×3 matrix, and $M_{\mathcal{G}}$ is the mass of the body, we have

$$\mathbf{B}_{\mathcal{G}} = M_{\mathcal{G}} \begin{Vmatrix} 0 & -\mathcal{R}_3 & \mathcal{R}_2 \\ \mathcal{R}_3 & 0 & -\mathcal{R}_1 \\ -\mathcal{R}_2 & \mathcal{R}_1 & 0 \end{Vmatrix} \tag{2.3}$$

The forces and moments acting on the body can be represented as

$$\mathbf{F} = \mathbf{F}_N + \mathbf{F}_L, \quad \mathbf{M} = \mathbf{M}_N + \mathbf{M}_L$$

where $\mathbf{F}_N$ and $\mathbf{M}_N$ are the force and moment due to the presence of the attracting centre, and $\mathbf{F}_L$ and $\mathbf{M}_L$ are the force and moment due to the presence of the fluid.

If $\mathcal{U}(\mathbf{x}, \mathbf{r})$ is the volume density of the gravitational forces, the Newtonian gravitational potential has the form

$$U_{\mathcal{G}}(r) = \int_{\mathcal{G}} \rho_{\mathcal{G}}(\mathbf{x})\mathcal{U}(\mathbf{x}, \mathbf{r})d\tau(\mathbf{x})$$

where $\rho_{\mathcal{G}}(\mathbf{x})$ is the density of the body. In this case

$$\mathbf{F}_N = -\frac{\partial U_{\mathcal{G}}}{\partial \mathbf{r}}, \quad \mathbf{M}_N = \mathbf{r} \times \frac{\partial U_{\mathcal{G}}}{\partial \mathbf{r}} \tag{2.4}$$

In the case considered

$$\mathcal{U}(\mathbf{x}, \mathbf{r}) = -f_N M_N |\mathbf{X}|^{-1}, \quad \mathbf{X} = \mathbf{x} + \mathbf{r} \tag{2.5}$$

The force and moment acting on the body from the fluid side have the form

$$\mathbf{F}_L = -\int_{\partial\mathcal{G}} p(\mathbf{x})\mathbf{n}(\mathbf{x})d\sigma(\mathbf{x}), \quad \mathbf{M}_L = -\int_{\partial\mathcal{G}} p(\mathbf{x})\mathbf{x} \times \mathbf{n}(\mathbf{x})d\sigma(\mathbf{x}) \tag{2.6}$$

where $p(\mathbf{x})$ is the fluid pressure at a point $\mathbf{x}$ on the body surface and $\mathbf{n}(\mathbf{x})$ is the vector of the unit outward normal at this point (see, for example, [3–5]).

Suppose the genus of $\partial\mathcal{G}$ is equal to zero and the fluid flow is potential. A unique function $\phi = \phi(\mathbf{x}, t)$ then exists which determines the fluid velocity field

$$\mathbf{v}(\mathbf{x}) = \partial\phi/\partial\mathbf{x} \tag{2.7}$$

and which satisfies Laplace's equation with boundary conditions

$$\Delta\phi = 0$$
$$\partial\phi/\partial\mathbf{n} = (\partial\phi/\partial\mathbf{x}, \mathbf{n}(\mathbf{x})) = (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{x}, \mathbf{n}(\mathbf{x})), \quad \mathbf{x} \in \partial\mathcal{G}$$
$$\partial\phi/\partial\mathbf{x} \to 0, \quad |\mathbf{x}| \to \infty$$

Hence, the determination of the potential of the fluid flow reduces to solving an external Neyman problem.

Note that if the genus of $\partial\mathcal{G}$ is greater than zero, we must seek a multivalue potential using, for example, the method of imaginary partitions. But, nevertheless, formulae (2.7) for determining the velocity field will again be applicable (see the details in [3, 4]).

After Neyman's problem has been solved, the pressure can be found from the Cauchy–Lagrange integral

$$p - p_0 = -\rho_L\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi}{\partial\mathbf{x}}, \frac{\partial\phi}{\partial\mathbf{x}}\right) + \mathcal{U}(x)\right) \tag{2.8}$$

The expressions for $\mathbf{F}_L$ and $\mathbf{M}_L$ can be represented in the form of sums

$$\mathbf{F}_L = \mathbf{F}_D + \mathbf{F}_S, \quad \mathbf{M}_L = \mathbf{M}_D + \mathbf{M}_S$$

The quantities with the subscripts D correspond to the first two terms on the right-hand side of the Cauchy-Lagrange integral and have a hydrodynamic origin, whereas the quantities with the subscript S, corresponding to the last term, have a hydrostatic origin; these forces and moments will henceforth be called Archimedes forces and moments.

By virtue of well-known discussions (see, for example, [5]), the hydrodynamic components of the forces and moments have the form

$$\mathbf{F}_D = -\frac{d\mathbf{K}}{dt} - \boldsymbol{\omega}\times\mathbf{K}, \quad M_D = -\frac{d\mathbf{L}}{dt} - \boldsymbol{\omega}\times\mathbf{L} - \mathbf{v}\times\mathbf{K}$$

where

$$\mathbf{K} = \mathbf{B}_L\boldsymbol{\omega} + \mathbf{C}_L\mathbf{v} = \frac{\partial T_L}{\partial\mathbf{v}}, \quad \mathbf{L} = \mathbf{A}_L\boldsymbol{\omega} + \mathbf{B}_L^T\mathbf{v} = \frac{\partial T_L}{\partial\boldsymbol{\omega}}$$

while the function

$$T_L = \frac{1}{2}((\mathbf{A}_L\boldsymbol{\omega}, \boldsymbol{\omega}) + 2(\mathbf{B}_L\boldsymbol{\omega}, \mathbf{v}) + (\mathbf{C}_L\mathbf{v}, \mathbf{v}))$$

is the kinetic energy of the fluid. The matrices $\mathbf{A}_L$, $\mathbf{B}_L$ and $\mathbf{C}_L$ define the tensor of the added masses of the body. Then, in the moving reference frame we have

$$T = T_C + T_L = \frac{1}{2}((\mathbf{A}\boldsymbol{\omega}, \boldsymbol{\omega}) + 2(\mathbf{B}\boldsymbol{\omega}, \mathbf{v}) + (\mathbf{C}\mathbf{v}, \mathbf{v}))$$

where the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are constants. The equation can then be represented in the form

$$\frac{d}{dt}\frac{\partial T}{\partial\boldsymbol{\omega}} = \frac{\partial T}{\partial\boldsymbol{\omega}}\times\boldsymbol{\omega} + \frac{\partial T}{\partial\mathbf{v}}\times\mathbf{v} + \mathbf{M}_E, \quad \frac{d}{dt}\frac{\partial T}{\partial\mathbf{v}} = \frac{\partial T}{\partial\mathbf{v}}\times\boldsymbol{\omega} + \mathbf{F}_E \tag{2.9}$$
$$\mathbf{F}_E = \mathbf{F}_N + \mathbf{F}_S, \quad \mathbf{M}_E = \mathbf{F}_N + \mathbf{F}_S$$

3. THE POTENTIAL ENERGY

By virtue of the Cauchy–Lagrange integral, the hydrostatic component of the pressure has the form $p_s = -\rho_L\mathcal{U}(x)$, where ρ_L is the fluid density. Hence, from formula (2.6) and Gauss' formula we have

$$\mathbf{F}_S = -\int_{\partial\mathcal{G}} p_S \mathbf{n} d\sigma(\mathbf{x}) = \rho_L \int_{\partial\mathcal{G}} \mathcal{U}(\mathbf{x})\mathbf{n} d\sigma(\mathbf{x}) = \rho_L \int_{\mathcal{G}} \frac{\partial \mathcal{U}}{\partial \mathbf{X}}(\mathbf{X}) d\tau(\mathbf{X})$$

In the case considered the only singularity in the integrand is outside the body. However

$$\frac{\partial}{\partial \mathbf{X}} \mathcal{U}(\mathbf{X}) = \frac{\partial}{\partial \mathbf{r}} \mathcal{U}(\mathbf{r}+\mathbf{x})$$

whence

$$\mathbf{F}_S = \frac{\partial}{\partial \mathbf{r}}\left[\rho_L \int_{\mathcal{G}} \mathcal{U}(\mathbf{r}+\mathbf{x}) d\tau(\mathbf{x})\right] = -\frac{\partial U_A}{\partial \mathbf{r}} \tag{3.1}$$

where

$$U_A = -\rho_L \int_{\mathcal{G}} \mathcal{U}(\mathbf{x}+r) d\tau(\mathbf{x}) \left(= f_N M_N \rho_L \int_{\mathcal{G}} |\mathbf{X}|^{-1} d\tau(\mathbf{x}) \right)$$

The moment of the hydrostatic (Archimedes) forces can be written in the form

$$\mathbf{M}_A = \mathbf{r} \times \frac{\partial U_A}{\partial \mathbf{r}} \tag{3.2}$$

Finally, the total potential has the form

$$U(\mathbf{r}) = U_N(\mathbf{r}) + U_A(\mathbf{r}) = -f_N M_N \int_{\mathcal{G}} |\mathbf{X}|^{-1} dm_a(\mathbf{x}) \tag{3.3}$$

where $dm_a(\mathbf{x}) = (\rho_{\mathcal{G}}(\mathbf{x}) - \rho_L) d\tau(\mathbf{x})$ is the distribution of the added masses of the body in the fluid.

Remark. If the body had performed motion in a region of finite dimensions ν, the expression for the potential energy could have been represented as

$$U(\mathbf{r}) = -f_N M_N \int_{\mathcal{G}} |\mathbf{X}|^{-1} dm_{\mathcal{G}}(\mathbf{x}) - f_N M_N \int_{\nu\setminus\mathcal{G}} |\mathbf{X}|^{-1} dm_L(\mathbf{x}) =$$

$$= -f_N M_N \int_{\mathcal{G}} |\mathbf{X}|^{-1} dm_{\mathcal{G}}(\mathbf{x}) + f_N M_N \int_{\mathcal{G}} |\mathbf{X}|^{-1} dm_L(\mathbf{x}) - f_N M_N \int_{\nu} |\mathbf{X}|^{-1} dm_L(\mathbf{x}) \tag{3.4}$$

If the potential density does not have singularities in the region ν, the last term is finite and is independent of $\mathbf{r}$. It makes no contribution to the expressions for the forces and moments. However, when the fluid fills the whole of space, these discussions can be regarded as a rigorous justification of the structure of the potential.

The equations of motion take the following explicit form

$$\frac{d}{dt}(\mathbf{A}\boldsymbol{\omega} + \mathbf{B}^T\mathbf{v}) = (\mathbf{A}\boldsymbol{\omega} + \mathbf{B}^T\mathbf{v}) \times \boldsymbol{\omega} + (\mathbf{B}\boldsymbol{\omega} + \mathbf{C}\mathbf{v}) \times \mathbf{v} - \frac{\partial U}{\partial \mathbf{r}} \times \mathbf{r} \tag{3.5}$$

$$\frac{d}{dt}(\mathbf{B}\boldsymbol{\omega} + \mathbf{C}\mathbf{v}) = (\mathbf{B}\boldsymbol{\omega} + \mathbf{C}\mathbf{v}) \times \boldsymbol{\omega} - \frac{\partial U}{\partial \mathbf{r}}$$

and they must be supplemented with kinematic equations (1.2).

4. APPROXIMATIONS FOR THE POTENTIAL

In Eqs (1.1) and (1.2) one can use both the exact expressions for the potential (3.3) and various approximations of them. These approximations may be related, for example, to the hypothesis that the body is small compared with the remoteness of the body from the attracting centre. In this case the quantities $\varepsilon(\mathbf{x}) = |\mathbf{x}|/r$, where $r = (r_1^2 + r_2^2 + r_3^2)^{1/2}$, are small compared with unity: $\sup_{x\in\mathscr{G}}|\varepsilon(\mathbf{x})| \ll 1$ and we can use the asymptotic expansion

$$\frac{1}{(\mathbf{r}+\mathbf{x},\mathbf{r}+\mathbf{x})^{1/2}} = \frac{1}{r}(1+u_1(\mathbf{r},\mathbf{x})+u_2(\mathbf{r},\mathbf{x}))+o\left(\frac{1}{r^3}\right)$$

to obtain approximate values of the potential. We have

$$u_1(\mathbf{r},\mathbf{x}) = -\frac{(\mathbf{r},\mathbf{x})}{r^2}, \quad u_2(\mathbf{r},\mathbf{x}) = \frac{\mathbf{x}^2}{r^2} - \frac{3}{2}\frac{(\mathbf{r}\times\mathbf{x})^2}{r^4}$$

Integrating the first relation over points of the body, we have

$$\int_{\mathscr{G}} u_1(\mathbf{r},\mathbf{x})dm_a(\mathbf{x}) = -\frac{1}{r^2}\left(\mathbf{r}, \int_{\mathscr{G}} \mathbf{x}dm_a(\mathbf{x})\right)$$

The integral on the right-hand side is related to the barycentre of the distribution of the added masses. More accurately, suppose $M_a = \int_{\mathscr{G}} dm_a(\mathbf{x})$ is the total added mass of the body (which can be both positive and negative or equal to zero) and C_a is the barycentre of the distribution of added masses, which exist, if $M_a \neq 0$. We have

$$\int_{\mathscr{G}} \mathbf{x}dm_a(\mathbf{x}) = \begin{cases} M_a\overrightarrow{CC_a} & \text{for } M_a \neq 0 \\ \boldsymbol{\mu} & \text{for } M_a = 0 \end{cases}$$

The vector μ, which is independent of the choice of the origin of coordinates and which only exists when $M_a = 0$, will be called the dipole moment. Finally we have

$$\int_{\mathscr{G}} u_1(\mathbf{r},\mathbf{x})dm_a(\mathbf{x}) = \begin{cases} -M_a\dfrac{1}{r^2}(\mathbf{r},\overrightarrow{CC_a}) & \text{for } M_a \neq 0 \\ -\dfrac{1}{r^2}(\mathbf{r},m) & \text{for } M_a = 0 \end{cases}$$

If $M_{\mathscr{G}}$ is the mass of the body, M_L is the mass of the fluid displaced by it, $C_{\mathscr{G}}$ is the centre of mass of the body and C_L is the centroid of this body, i.e. the centre of the mass of the fluid displaced by the body, we have $M_a = M_{\mathscr{G}} - M_L$ and

$$M_a\overrightarrow{CC_a} = M_{\mathscr{G}}\overrightarrow{CC_{\mathscr{G}}} - M\overrightarrow{CC_L} \text{ for } M_a \neq 0, \quad \boldsymbol{\mu} = M_{\mathscr{G}}\overrightarrow{C_LC_{\mathscr{G}}} \text{ for } M_a = 0$$

Integration of the terms of the second order of smallness gives

$$\int_{\mathscr{G}} u_2(\mathbf{r},\mathbf{x})dm_a(\mathbf{x}) = \frac{3}{2}\frac{1}{r^4}\int_{\mathscr{G}}\left\{\frac{2}{3}\mathbf{r}^2\mathbf{x}^2 - (\mathbf{r}\times\mathbf{x})^2\right\}dm_a(\mathbf{x}) = -\frac{3}{2}\frac{1}{r^4}\mathbf{I}_a^*(\mathbf{r},\mathbf{r})$$

$$\mathbf{I}_a^* = \mathbf{I}_a - \frac{1}{3}\mathrm{Tr}(\mathbf{I}_a)\mathbf{E}$$

($\mathbf{I}_a^*$ is the deviator of the inertia tensor of the added masses $\mathbf{I}_a$).

Finally, the expression for the potential has the form

$$U(\mathbf{r}) = U_0(\mathbf{r}) + U_1(\mathbf{r}) + U_2(\mathbf{r}) + o(1/r^3) \tag{4.1}$$

$$U_0(\mathbf{r}) = -f_N M_N \frac{M_a}{r}, \quad U_2(\mathbf{r}) = \frac{\varkappa}{2}\frac{(\mathbf{D}\mathbf{r},\mathbf{r})}{r^5}$$

$$U_1(\mathbf{r}) = \begin{cases} f_N M_N M_a \dfrac{(\mathbf{r}, \overrightarrow{CC_a})}{r^3} & \text{for } M_a \neq 0 \\ f_N M_N \dfrac{(\mathbf{r}, \boldsymbol{\mu})}{r^3} & \text{for } M_a = 0 \end{cases}$$

($\mathbf{D} = 3\mathbf{I}_a^*$ is a tensor with zero trace and $\varkappa$ is a constant). Hence, we obtain two very different cases.

The case $M_a \neq 0$. Choosing the point C_a as the origin of the moving reference frame, the potential can be reduced to the form

$$U(\mathbf{r}) = U_0(\mathbf{r}) + U_2(\mathbf{r}) + o\left(\frac{1}{r^3}\right) = -f_N M_N \frac{M_a}{r} + \frac{\varkappa}{2} \frac{(\mathbf{Dr}, \mathbf{r})}{r^5} + \ldots$$

In the first approximation the potential is equal to U_0, and this potential generates an attractive or repulsive force directed towards the centre N or from the centre N depending on whether the added mass M_a is positive or negative. Note that the potential of the first approximation only affects the translational motions of the body and has no effect on its rotation. The component U_2 is decisive for determining the orientation of the body.

The case $M_a = 0$. The potential takes the form

$$U(\mathbf{r}) = U_1(\mathbf{r}) + U_2(\mathbf{r}) + o\left(\frac{1}{r^3}\right) = f_N M_N \frac{(\mathbf{r}, \boldsymbol{\mu})}{r^3} + \frac{\varkappa}{2} \frac{(\mathbf{Dr}, \mathbf{r})}{k^5} + \ldots$$

In the first approximation, therefore, the situation is neutral. However, in the next approximation the component U_1 has a decisive effect both on the translational motion of the body and on its rotation. Such a situation does not occur in the orbital dynamics of rigid bodies under gravity forces, so that this case deserves a separate consideration. We merely note that it only occurs for a non-uniform body, since it follows from the fact that the added mass is equal to zero in the case of a uniform body that the densities of the body and the fluid are the same and, as a consequence, the potential $U(\mathbf{r})$, is identically equal to zero.

5. THE SATELLITE APPROXIMATION

In the orbital mechanics of rigid and deformable bodies the so-called "satellite approximation" is well known. It enables the problem to be simplified and also to distinguish between the motion of the centre of mass of the system and its motion around the centre of mass. The question arises of whether one can indicate those values of the parameters in the problem of the motion of a body in a fluid for which the analogue of the satellite approximation can be used.

We will consider the problem of the smallness of the body as follows. We will assume that a family of bodies exists, homothetic between one another, with a common homothety centre at the point C and that this family is parametrized with a homothety ratio ε. Suppose

$$f(x_1/\varepsilon, x_2/\varepsilon, x_3/\varepsilon) = 0$$

is the parametric equation of the surfaces of these bodies. If $\varepsilon \to 0$, the dimensions of the body also tend to zero. The body density is assumed to be independent of the dimensions of the body, i.e. of the parameter ε. Then the solution of Laplace's equation can be expressed as a function of ε. Substituting its solution into the formulae for the components of the kinetic energy matrix, we obtain their dependence on this parameter. We have

$$M_k(\varepsilon) = \varepsilon^3 M_k(1), \;\; k \in \{\mathcal{G}, L\}, \;\; B_{\mathcal{G}}(\varepsilon) = \varepsilon^4 B_{\mathcal{G}}(1), \;\; I_{\mathcal{G}}(\varepsilon) = \varepsilon^5 I_{\mathcal{G}}(1)$$
$$\mathbf{A}_L(\varepsilon) = \varepsilon^5 \mathbf{A}_L(1), \;\; \mathbf{B}_L(\varepsilon) = \varepsilon^4 \mathbf{B}_L(1), \;\; \mathbf{C}_L(\varepsilon) = \varepsilon^3 \mathbf{C}_L(1)$$

Then

$$\mathbf{A}(\varepsilon) = \varepsilon^5 \mathbf{A}(1), \;\; \mathbf{B}(\varepsilon) = \varepsilon^4 \mathbf{B}(1), \;\; \mathbf{C}(\varepsilon) = \varepsilon^2 \mathbf{C}(1) \tag{5.1}$$

From the same considerations

$$\mathbf{D}(\varepsilon) = \varepsilon^5 \mathbf{D}(1) \tag{5.2}$$

$$U(\mathbf{r}, \varepsilon) = U_0(\mathbf{r}, \varepsilon) + U_1(\mathbf{r}, \varepsilon) + U_2(\mathbf{r}, \varepsilon) + \ldots = \varepsilon^3 U_0(\mathbf{r}, 1) + \varepsilon^4 U_1(\mathbf{r}, 1) + \varepsilon^5 U_2(\mathbf{r}, 1) + \ldots \tag{5.3}$$

Consider, for example, a body in the form of an ellipsoid. Suppose its centre and principal axes coincide with the centre and principal axes, respectively, of a moving reference frame, while the semi-axes a_i depend on the parameter ε

$$\alpha_i = \varepsilon a_i', \quad a_i' = O(1) \tag{5.4}$$

The structure of the components of the tensor of the added masses is well known [4]. We have

$$\mathbf{C}_L = \operatorname{diag}(C_{L1}, C_{L2}, C_{L3}), \quad C_{Li} = \frac{\delta_i}{2-\delta_i}\rho_L \nu, \quad i = 1,2,3 \tag{5.5}$$

$$\delta_i(a_1, a_2, a_3) = a_1 a_2 a_3 \int_0^\infty \frac{d\lambda}{(a_i^2+\lambda)\Delta} \quad \Delta = ((a_1^2+\lambda)(a_2^2+\lambda)(a_3^2+\lambda))^{1/2}$$

($\nu = \frac{4}{3}\pi a_1 a_2 a_3$ is the volume of the body). The matrix $\mathbf{B}_L$ is equal to zero. The matrix $\mathbf{A}_L$ has the form

$$\mathbf{A}_L = \operatorname{diag}(A_{L1}, A_{L2}, A_{L3}), \quad A_{L1} = \frac{1}{5}\frac{(a_2^2-a_3^2)^2(\delta_3-\delta_2)}{2(a_2^2-a_3^2)+(a_2^2+a_3^2)(\delta_2-\delta_3)}\rho_L \nu \ (1\ 2\ 3)$$

Substituting expressions (5.4) and $\lambda = \lambda'\varepsilon^2$ into (5.5) and assuming that the fluid density ρ_L is constant, we have

$$\delta_i(a_1, a_2, a_3) = \delta_i(a_1', a_2', a_3')$$

$$C_{Li}(a_1, a_2, a_3) = \varepsilon^3 C_{Li}(a_1', a_2', a_3'), \quad A_{Li}(a_1, a_2, a_3) = \varepsilon^5 A_{Li}(a_1', a_2', a_3')$$

Suppose $\varepsilon \neq 0$. Then, substituting (5.1) and (5.3) into the equations of motion, separating their left-hand and right-hand sides in $\varepsilon^2 \neq 0$ and discarding the argument of unity in the matrices, we have

$$\frac{d}{dt}(\varepsilon^2\mathbf{A}\boldsymbol{\omega} + \varepsilon\mathbf{B}^T\mathbf{v}) = (\varepsilon^2\mathbf{A}\boldsymbol{\omega} + \varepsilon\mathbf{B}^T\mathbf{v})\times\boldsymbol{\omega} + (\varepsilon\mathbf{B}\boldsymbol{\omega} + \mathbf{C}\mathbf{v})\times\mathbf{v} - \left(\varepsilon\frac{\partial U_1}{\partial \mathbf{r}} + \varepsilon^2\frac{\partial U_2}{\partial \mathbf{r}} + \ldots\right)\times\mathbf{r} \tag{5.6}$$

$$\frac{d}{dt}(\varepsilon\mathbf{B}\boldsymbol{\omega} + \mathbf{C}\mathbf{v}) = (\varepsilon\mathbf{B}\boldsymbol{\omega} + \mathbf{C}\mathbf{v})\times\boldsymbol{\omega} - \left(\frac{\partial U_0}{\partial \mathbf{r}} + \varepsilon\frac{\partial U_1}{\partial \mathbf{r}} + \ldots\right) \tag{5.7}$$

We will seek solutions in the form of formal series

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \varepsilon\boldsymbol{\omega}_1 + \ldots, \quad \mathbf{v} = \mathbf{v}_0 + \varepsilon\mathbf{v}_1 + \ldots, \quad \mathbf{r} = \mathbf{r}_0 + \varepsilon\mathbf{r}_1 + \ldots$$

Since further discussions only touch on terms of the lowest order, the subscript zero will henceforth be omitted everywhere.

We will assume that the tensor $\mathbf{C}$ is non-spherical. Then, in the first group of equations (5.6) the parameter ε^2 remains a factor for the higher derivative and a direct analogy with the satellite approximation does not exist in the general case. Possible methods of overcoming this difficulty will not be considered in this paper.

We will now assume that the tensor $\mathbf{C}$ is spherical: $\mathbf{C} = c\mathbf{E}$. Then $\mathbf{C}\mathbf{v} \times \mathbf{v} = 0$ and both sides of Eqs (5.6) can be separated for $\varepsilon \neq 0$. In the limit as $\varepsilon \to 0$

$$\frac{d}{dt}\mathbf{B}^T\mathbf{v} = \mathbf{B}^T\mathbf{v}\times\boldsymbol{\omega} + \mathbf{B}\boldsymbol{\omega}\times\mathbf{v} - \frac{\partial U_1}{\partial \mathbf{r}}\times\mathbf{r} \tag{5.8}$$

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega} - \frac{1}{c}\frac{\partial U_0}{\partial \mathbf{r}} \tag{5.9}$$

We will write Eqs (5.9) with respect to the absolute system of coordinates. In them $\mathbf{R}$ is the vector $\overrightarrow{NC}$, and the equations have the form

$$\frac{d}{dt}\frac{\partial L_0}{\partial \dot{\mathbf{R}}} = \frac{\partial L_0}{\partial \mathbf{R}} \tag{5.10}$$

with Lagrangian

$$L_0(\dot{\mathbf{R}}, \mathbf{R}) = \frac{c}{2}\dot{\mathbf{R}}^2 + f_N M_N (M_{\mathscr{G}} - M_L)\frac{1}{r}, \quad r = (\mathbf{r}, \mathbf{r})^{1/2} = (\mathbf{R}, \mathbf{R})^{1/2}$$

In explicit form they can be written as

$$\frac{d^2\mathbf{R}}{dt^2} = -\frac{1}{c} f_N M_N (M_{\mathscr{G}} - M_L)\frac{\mathbf{R}}{r^3} \tag{5.11}$$

These equations can be integrated in the same way as the equations in Kepler's problem, but the qualitative properties of the motion under certain conditions turn out to be quite different. For example, if $M_L > M_{\mathscr{G}}$ there are no closed orbits. If $M_L = M_{\mathscr{G}}$, the point C in this approximation moves with constant velocity along a straight line, which does not necessarily pass through the origin of coordinates, but is not observed in Kepler's problem. Finally, if $M_L < M_{\mathscr{G}}$, then, as in Kepler's problem, there are hyperbolic, parabolic and elliptic orbits. But the parameters of these orbits depend both on the masses $M_{\mathscr{G}}$ and M_L and on the coefficient c of the tensor of the added masses.

After Eqs (5.11) have been integrated, i.e. the relations

$$\mathbf{R} = \mathbf{R}(t), \quad \mathbf{V} = \dot{\mathbf{R}}(t) = \mathbf{V}(t)$$

have been obtained, the quantities $\mathbf{r}$ and $\mathbf{v}$ can be represented as functions of time and of the orientation of the body

$$\mathbf{r} = \mathbf{S}^T\mathbf{R}(t), \quad \mathbf{v} = \mathbf{S}^T\mathbf{V}(t), \quad \mathbf{S} = \begin{Vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{Vmatrix} \tag{5.12}$$

Substituting these expressions into Eqs (5.8) we obtain a system of algebraic equations in ω. When the conditions which ensure compatibility of the system and the uniqueness of its solution are satisfied, substitution of this equation into the system of Poisson's equations gives a closed system for determining the change in the orientation of the body. We will not dwell on the details here.

For the reasons indicated above, the gravitational moment plays no decisive role in system (5.8), except when $M_{\mathscr{G}} = M_L$. But in this case, by virtue of Eqs (5.11), the point C moves along a straight line and the problem cannot be regarded as the problem of the motion of a satellite. If $M_{\mathscr{G}} \neq M_L$, the point C moves in a Kepler orbit, but, in the approximation considered, the attraction forces play no role in the motion of the body around the point C.

Finally, suppose $\mathbf{C} = c\mathbf{E}$, $\mathbf{B} = 0$ and $M_{\mathscr{G}} \neq M_L$. Then, after dividing Eq. (5.6) by $\varepsilon^2 \neq 0$, we obtain the equation

$$\frac{d}{dt}\mathbf{A}\boldsymbol{\omega} = \mathbf{A}\boldsymbol{\omega} \times \boldsymbol{\omega} - \frac{\partial U_2}{\partial \mathbf{r}} \times \mathbf{r} = \mathbf{A}\boldsymbol{\omega} \times \boldsymbol{\omega} - \frac{\varkappa}{r^5}\mathbf{D}\mathbf{r} \times \mathbf{r} \tag{5.13}$$

which together with Poisson's equations describes the change in the orientation of the body. In the general case these equations are again the Euler–Lagrange–Poincaré equations

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \boldsymbol{\omega}} = \frac{\partial \mathscr{L}}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial \mathscr{L}}{\partial \mathbf{r}} \times \mathbf{r}, \quad \mathscr{L}(\boldsymbol{\omega}, \mathbf{r}) = \frac{1}{2}(\mathbf{A}\boldsymbol{\omega}, \boldsymbol{\omega}) - U_2(\mathbf{r}) \tag{5.14}$$

The motion of the point C is again given by Eq. (5.11). Substituting $\mathbf{r} = \mathrm{S}^T\mathbf{R}(t)$ from (5.12) into (5.13) and considering the latter together with Poisson's equations, we have a closed, in general, non-autonomous system of 12 equations in 12 unknowns for determining the orientation of the body and the change in angular velocity.

By analogy with Kepler's problem we conclude that in the approximation considered the point C moves in a plane fixed in absolute space and perpendicular to the components of the angular momentum vector corresponding to Eqs (5.11). By virtue of the integrals (1.8) this plane coincides with the $NX_\alpha X_\gamma$ plane. In order to integrate Eqs (5.11) we will assume that the point C moves in the same plane. In this plane we introduce the polar coordinates

$$R_\gamma = r\cos\psi, \quad R_\alpha = r\sin\psi$$

Lagrange's function (5.10) can then be written in the form

$$L(\dot{r},\ \dot{\psi},\ r) = \frac{c}{2}(\dot{r}^2 + r^2\dot{\psi}^2) + f_N M_N (M_{\mathscr{G}} - M_L)\frac{1}{r}$$

The coordinate ψ is cyclic, and the corresponding first integral has the form

$$cr^2\dot{\psi} = p_\psi \tag{5.15}$$

whence

$$\dot{\psi} = p_\psi/(cr^2)$$

and Routh's function can be represented as

$$R(\dot{r},\ r,\ p_\psi) = c\dot{r}^2/2 - U_A(r,\ p_\psi), \quad U_A(r,\ p_\psi) = p_\psi^2/(2cr^2) + U(r) \tag{5.16}$$

where $U_A(r, p_\psi)$ is the reduced potential. The critical points of this potential correspond to the radii of circular orbits. We have

$$\frac{\partial U_A}{\partial r} = -\frac{p_\psi^2}{cr^3} + f_N M_N (M_{\mathscr{G}} - M_L)\frac{1}{r^2} = 0$$

whence we derive that in these orbits

$$r = \frac{p_\psi^2}{cf_N M_N (M_{\mathscr{G}} - M_L)} = \left(\frac{f_N M_N (M_{\mathscr{G}} - M_L)}{c\dot{\psi}^2}\right)^{1/3} \tag{5.17}$$

or in the "Kepler's law" form

$$r^3\dot{\psi}^2 = f_N M_N (M_{\mathscr{G}} - M_L)/c$$

Hence the relation between the radius of the orbit, the masses and the added masses follows.

Since $c \geqslant M_{\mathscr{G}}$, while $M_L \geqslant 0$, in general the constant on the right-hand side of the last equation is less than the constant on the right-hand side in the case when there is no fluid. In other words, for this orbital radius, the orbital angular velocity of the motion in a space filled with fluid is less than the orbital velocity in a vacuum.

We can use the true anomaly ψ as the independent variable instead of the time. In the case of an elliptic orbit this replacement enables us to find the equation of the orbit

$$r = p/(1 - e\cos\psi) \tag{5.18}$$

where p is the parameter of the ellipse and e is its eccentricity. Substituting (5.18) into (5.15) we obtain an equation which determines ψ as a function of time. With the exception of the case of circular motion, the solution of this equation, called Kepler's equation, cannot be expressed in a final form. In order to avoid solving it in explicit form, when describing the change in the orientation of the body we also use the true anomaly as the independent variable.

6. THE DYNAMICS OF THE SYSTEM WITH RESPECT TO A UNIFORMLY ROTATING "ORBITAL" SYSTEM OF COORDINATES

We will now consider an orbital system of coordinates $NX'_\alpha X'_\beta X'_\gamma$, rotating around the NX'_β axis, coinciding with the NX_β axis. Suppose NX'_γ is the axis directed along the vector $\overrightarrow{NC}$, the axis NX'_β is orthogonal to the orbital plane, the axis NX'_α is in the orbital plane, and we supplement NX'_γ and NX'_β up to the right triple. Suppose

$$\boldsymbol{\alpha}' = (\alpha'_1,\ \alpha'_2,\ \alpha'_3),\quad \boldsymbol{\beta}' = (\beta'_1,\ \beta'_2,\ \beta'_3),\quad \boldsymbol{\gamma}' = (\gamma'_1,\ \gamma'_2,\ \gamma'_3)$$

are unit vectors of this system of coordinates. Then

$$\boldsymbol{\alpha}' = \boldsymbol{\alpha}\cos\psi - \boldsymbol{\gamma}\sin\psi,\quad \boldsymbol{\beta}' = \boldsymbol{\beta},\quad \boldsymbol{\gamma}' = \boldsymbol{\alpha}\sin\psi + \boldsymbol{\gamma}\cos\psi$$

In general this reference frame rotates non-uniformly. We will confine ourselves solely to the case of circular motion, in which case the orbital angular velocity $\dot\psi$ = const. Suppose Ω is the angular velocity relative to the reference frame $NX'_\alpha X'_\beta X'_\gamma$. This velocity and the absolute angular velocity ω are related by the expression

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \dot\psi\boldsymbol{\beta} \tag{6.1}$$

We will put

$$\mathcal{L}_r(\boldsymbol{\Omega},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) = \mathcal{L}(\boldsymbol{\Omega} + \dot\psi\boldsymbol{\beta},\ r\boldsymbol{\gamma})$$

Here and henceforth in this and the following sections the primes will be omitted everywhere.

We have

$$\frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\Omega}} \equiv \frac{\partial\mathcal{L}}{\partial\boldsymbol{\omega}},\quad \frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\beta}} = \dot\psi\frac{\partial\mathcal{L}}{\partial\boldsymbol{\omega}},\quad \frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\gamma}} = r\frac{\partial\mathcal{L}}{\partial\mathbf{r}}$$

Hence, by virtue of (5.14)

$$\frac{d}{dt}\frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\Omega}} = \frac{\partial\mathcal{L}}{\partial\boldsymbol{\omega}}\times(\boldsymbol{\Omega}+\dot\psi\boldsymbol{\beta}) + \frac{1}{r}\frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\gamma}}\times(r\boldsymbol{\gamma}) = \frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\Omega}}\times\boldsymbol{\Omega} + \frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\beta}}\times\boldsymbol{\beta} + \frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\gamma}}\times\boldsymbol{\gamma} \tag{6.2}$$

We will write Lagrange's function for the problem in question

$$\mathcal{L}_r = \frac{1}{2}(\mathbf{A}(\boldsymbol{\Omega}+\dot\psi\boldsymbol{\beta}),\ \boldsymbol{\Omega}+\dot\psi\boldsymbol{\beta}) - \frac{\varkappa}{r^3}(\mathbf{D}\boldsymbol{\gamma},\ \boldsymbol{\gamma}) =$$

$$= \frac{1}{2}(\mathbf{A}\boldsymbol{\Omega},\ \boldsymbol{\Omega}) + \dot\psi(\mathbf{A}\boldsymbol{\Omega},\ \boldsymbol{\beta}) + \frac{1}{2}\dot\psi^2(\mathbf{A}\boldsymbol{\beta},\ \boldsymbol{\beta}) - \frac{1}{2}K(\mathbf{D}\boldsymbol{\gamma},\ \boldsymbol{\gamma}),\quad K = \frac{\varkappa}{r^3} \tag{6.3}$$

By virtue of Poisson's equation for the reference frame $NX'_\alpha X'_\beta X'_\gamma$

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\times\boldsymbol{\Omega},\quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta}\times\boldsymbol{\Omega},\quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\Omega} \tag{6.4}$$

the Euler–Lagrange–Poincaré equations take the form

$$\mathbf{A}\dot{\boldsymbol{\Omega}} = \mathbf{A}\boldsymbol{\Omega}\times\boldsymbol{\Omega} + \dot\psi[\mathbf{A}\boldsymbol{\beta}\times\boldsymbol{\Omega} + \mathbf{A}\boldsymbol{\Omega}\times\boldsymbol{\beta} - \mathbf{A}(\boldsymbol{\beta}\times\boldsymbol{\Omega})] + \dot\psi^2\mathbf{A}\boldsymbol{\beta}\times\boldsymbol{\beta} + K\mathbf{D}\boldsymbol{\gamma}\times\boldsymbol{\gamma} \tag{6.5}$$

In addition to the geometrical integrals

$$\mathcal{F}_{ii} = (\mathbf{i},\ \mathbf{i}) - 1 = 0,\quad \mathbf{i}\in\{\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}\} \tag{6.6}$$

$$\mathcal{F}_{ij} = (\mathbf{i},\ \mathbf{j}) = 0,\quad \mathbf{i},\ \mathbf{j}\in\{\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}\},\quad \mathbf{i}\neq\mathbf{j}$$

system (6.4) and (6.5) allows of the (Penlevé–Jacobi) energy integral

$$\mathcal{F}_l = \left(\frac{\partial\mathcal{L}_r}{\partial\boldsymbol{\Omega}},\ \boldsymbol{\Omega}\right) - \mathcal{L}_r = \frac{1}{2}(\mathbf{A}\boldsymbol{\Omega},\ \boldsymbol{\Omega}) + \frac{1}{2}[K(\mathbf{D}\boldsymbol{\gamma},\ \boldsymbol{\gamma}) - \dot\psi^2(\mathbf{A}\boldsymbol{\beta},\ \boldsymbol{\beta})] \tag{6.7}$$

The function

$$U_a(\boldsymbol{\beta},\ \boldsymbol{\gamma}) = -\mathcal{L}_r(0,\ \boldsymbol{\beta},\ \boldsymbol{\gamma}) = -\dot{\psi}^2(\mathbf{A}\boldsymbol{\beta},\ \boldsymbol{\beta})/2 + K(\mathbf{D}\boldsymbol{\gamma},\ \boldsymbol{\gamma})/2 \tag{6.8}$$

is called the augmented potential of the system considered.

For complete integrability of the equations of the satellite approximation in the case of a circular orbit we lack two additional commutative independent first integrals.

7. RELATIVE EQUILIBRIA WITHIN THE FRAMEWORK OF THE SATELLITE APPROXIMATION

Using Routh's method we can obtain the steady motions and investigate the sufficient conditions for their stability in the satellite approximation. Consider the Routh function

$$W = \mathcal{F}_I + \lambda\mathcal{F}_\beta/2 + \mu\mathcal{F}_\gamma/2 + \nu\mathcal{F}_{\beta\gamma} \tag{7.1}$$

and put

$$\tau = r^3\dot{\psi}^2/\varkappa = \dot{\psi}^2/K$$

Since the vector α does not occur explicitly in Lagrange's function, the integrals with α are not included in this linear combination. The critical points of function (7.1) correspond to steady motions of the system considered and can be found from the equations

$$\frac{\partial W}{\partial \boldsymbol{\Omega}} = \frac{\partial^2 \mathcal{L}_r}{\partial \boldsymbol{\Omega}^2}\boldsymbol{\Omega} + \frac{\partial \mathcal{L}_r}{\partial \boldsymbol{\Omega}} - \frac{\partial \mathcal{L}_r}{\partial \boldsymbol{\Omega}} = \frac{\partial^2 \mathcal{L}_r}{\partial \boldsymbol{\Omega}^2}\boldsymbol{\Omega} = \mathbf{A}\boldsymbol{\Omega} = 0 \tag{7.2}$$

$$\frac{\partial W}{\partial \boldsymbol{\beta}} = \frac{\partial^2 \mathcal{L}_r}{\partial \boldsymbol{\beta}\partial \boldsymbol{\Omega}}\boldsymbol{\Omega} - \frac{\partial \mathcal{L}_r}{\partial \boldsymbol{\beta}} + \lambda\boldsymbol{\beta} + \nu\boldsymbol{\gamma} = 0 \tag{7.3}$$

$$\frac{\partial W}{\partial \boldsymbol{\gamma}} = \frac{\partial^2 \mathcal{L}_r}{\partial \boldsymbol{\gamma}\partial \boldsymbol{\Omega}}\boldsymbol{\Omega} - \frac{\partial \mathcal{L}_r}{\partial \boldsymbol{\gamma}} + \nu\boldsymbol{\beta} + \mu\boldsymbol{\gamma} = 0 \tag{7.4}$$

To determine the relative equilibria we must consider these equations together with the integrals from (6.6), which are independent of α, as a system in Ω, β, γ and Lagrange multipliers λ, μ and ν.

Equation (7.2) always allows of the solution $\Omega = 0$, and if the matrix $\partial^2\mathcal{L}_r/\partial\Omega^2 = \mathbf{A}$ is non-degenerate, which happens in mechanics, this solution is unique. From the point of view of mechanics the equality $\Omega = 0$ denotes that these motions are steady, i.e. the rigid body is immobile in the orbital system of coordinates $NX'_\alpha X'_\beta X'_\gamma$. In other words, in these motions the system is in equilibrium with respect to a reference frame uniformly rotating around the NX_β axis, and these motions in this case are relative equilibria.

From Eqs (7.3) and (7.4) we obtain

$$(\lambda\mathbf{E} - \dot{\psi}^2\mathbf{A})\boldsymbol{\beta} + \nu\boldsymbol{\gamma} = 0 \tag{7.5}$$

$$\nu\boldsymbol{\beta} + (K\mathbf{D} + \mu\mathbf{E})\boldsymbol{\gamma} = 0 \tag{7.6}$$

Equations (7.5) and (7.6) have a more general form than the classical equations describing the relative equilibria of a satellite moving in a vacuum (see, for example, [1, 2, 6]).

Multiplying Eq. (7.5) scalarly by γ and Eq. (7.6) by β and using geometrical integrals, we obtain

$$\nu = \dot{\psi}^2(\mathbf{A}\boldsymbol{\beta},\ \boldsymbol{\gamma}) = -K(\mathbf{D}\boldsymbol{\beta},\ \boldsymbol{\gamma}) \tag{7.7}$$

Multiplying Eq. (7.5) scalarly by β and Eq. (7.6) by γ, we obtain

$$\lambda = \dot{\psi}^2(\mathbf{A}\boldsymbol{\beta},\ \boldsymbol{\beta}),\quad \mu = -K(\mathbf{D}\boldsymbol{\gamma},\ \boldsymbol{\gamma}) \tag{7.8}$$

Nevertheless, by eliminating the Lagrange multipliers in this way we cannot obtain a closed subsystem of equations in β or γ. However, we can use the method described previously in [6]. Rewriting Eq. (7.5), we obtain

$$(\dot\psi^2\mathbf{A}-\lambda\mathbf{E})\boldsymbol\beta=\nu\boldsymbol\gamma$$

Scalar multiplication of the left-hand and right-hand sides, the uniqueness of β and γ and the expression for λ from (7.8) imply the relations

$$\begin{aligned}\nu&=\varepsilon_1((\dot\psi^2\mathbf{A}-\lambda\mathbf{E})\boldsymbol\beta,\ (\dot\psi^2\mathbf{A}-\lambda\mathbf{E})\boldsymbol\beta)^{1/2}=\varepsilon_1(\lambda^2-2\lambda\dot\psi^2(\mathbf{A}\boldsymbol\beta,\ \boldsymbol\beta)+\dot\psi^4(\mathbf{A}\boldsymbol\beta,\ \mathbf{A}\boldsymbol\beta))^{1/2}=\\&=\varepsilon_1\dot\psi^2((\mathbf{A}\boldsymbol\beta,\ \mathbf{A}\boldsymbol\beta)-(\mathbf{A}\boldsymbol\beta,\ \boldsymbol\beta)^2)^{1/2}=\\&=\varepsilon_1\dot\psi^2(\mathbf{A}\boldsymbol\beta\times\boldsymbol\beta,\ \mathbf{A}\boldsymbol\beta\times\boldsymbol\beta)^{1/2}=\varepsilon_1\dot\psi^2\,|\mathbf{A}\boldsymbol\beta\times\boldsymbol\beta|,\ \ \varepsilon_1=\pm1\end{aligned}\tag{7.9}$$

Then, by virtue of the same equation (7.5)

$$\boldsymbol\gamma=\varepsilon_1(\mathbf{A}-(\mathbf{A}\boldsymbol\beta,\ \boldsymbol\beta)\mathbf{E})\boldsymbol\beta\,|\mathbf{A}\boldsymbol\beta\times\boldsymbol\beta|^{-1}=-\varepsilon_1\boldsymbol\beta\times(\boldsymbol\beta\times\mathbf{A}\boldsymbol\beta)\,|\mathbf{A}\boldsymbol\beta\times\boldsymbol\beta|^{-1}\tag{7.10}$$

if the vector $A\beta$ is not collinear with the vector β or if $\nu \neq 0$.

Using Eq. (7.6) we find in the same way that

$$\nu=\varepsilon_2K|\mathbf{D}\boldsymbol\gamma\times\boldsymbol\gamma|,\quad \varepsilon_2=\pm1\tag{7.11}$$

$$\boldsymbol\beta=\varepsilon_2\,\mathrm{sign}\,K((\mathbf{D}\boldsymbol\gamma,\ \boldsymbol\gamma)\mathbf{E}-\mathbf{D})\boldsymbol\gamma\,|\mathbf{D}\boldsymbol\gamma\times\boldsymbol\gamma|^{-1}=-\varepsilon_2\,\mathrm{sign}\,K\boldsymbol\gamma\times(\mathbf{D}\boldsymbol\gamma\times\boldsymbol\gamma)\,|\mathbf{D}\boldsymbol\gamma\times\boldsymbol\gamma|^{-1}\tag{7.12}$$

Substituting the expression for β from the first equation of (7.12) into (7.10), we obtain the equation

$$(\mathcal{B}(\boldsymbol\gamma)-\mathcal{C}(\boldsymbol\gamma))\boldsymbol\gamma=0$$

Hence, the vector γ must belong to the kernel of the matrix $(\mathcal{B}(\gamma)-\mathcal{C}(\gamma))$ and the existence of a solution implies the equality $\det(\mathcal{B}(\gamma)-\mathcal{C}(\gamma))=0$, which again indicates that the solutions are situated on a certain surface in γ space. Moreover, these solutions are situated at the intersection of this surface with the sphere $\gamma^2-1=0$. These conditions can also be obtained in the space of the variables β.

8. INVESTIGATION OF THE RELATIVE EQUILIBRIA USING THE EQUATIONS OF RELATIVE MOTION

The relative equilibria can, of course, also be obtained directly from the equations of relative motion. Assuming $\Omega=0$ in Eqs (6.5), we obtain the system of algebraic equations

$$\dot\psi^2\mathbf{A}\boldsymbol\beta\times\boldsymbol\beta=K\mathbf{D}\boldsymbol\gamma\times\boldsymbol\gamma\tag{8.1}$$

which express the equality of the moments of the active and centrifugal forces. The projection of these moments onto the axes of an orbital system of coordinates give the system of equations

$$-\dot\psi^2(\mathbf{A}\boldsymbol\beta,\ \boldsymbol\gamma)=K(\mathbf{D}\boldsymbol\gamma,\ \boldsymbol\beta),\quad 0=-K(\mathbf{D}\boldsymbol\gamma,\ \boldsymbol\alpha),\quad \dot\psi^2(\mathbf{A}\boldsymbol\beta,\ \boldsymbol\alpha)=0\tag{8.2}$$

which must also be considered together with integrals (6.6).

Note that the first of equations (8.2) is equivalent to Eq. (7.7).

Using the geometrical relations $\alpha=\beta\times\gamma$, we eliminate α from Eqs (8.2). Now, instead of all the integrals (6.6) it is sufficient to consider only those which do not depend on α. Separating the homogeneous and non-homogeneous subsystems from the system obtained, we finally have

$$(\boldsymbol\beta,\boldsymbol\gamma)=0,\ (\mathbf{D}\boldsymbol\gamma,\boldsymbol\beta\times\boldsymbol\gamma)=0,\ (\mathbf{A}\boldsymbol\beta,\boldsymbol\beta\times\boldsymbol\gamma)=0,\ \tau(\mathbf{A}\boldsymbol\beta,\boldsymbol\gamma)+(\mathbf{D}\boldsymbol\gamma,\boldsymbol\beta)=0\tag{8.3}$$

$$(\boldsymbol\beta,\boldsymbol\beta)=1,\ \ (\boldsymbol\gamma,\boldsymbol\gamma)=1\tag{8.4}$$

Hence, if we succeed in obtaining a certain-non-zero solution of the homogeneous subsystem (8.3), then, using the non-homogeneous subsystem (8.4), we can normalize this solution.

By virtue of Eqs (7.5) (correspondingly (7.6)) the vector $\mathbf{A}\beta$ (correspondingly the vector $\mathbf{D}\gamma$) must lie in the plane generated by the vectors β and γ, for which it is necessary and sufficient that the vector $\mathbf{A}\beta$ (correspondingly $\mathbf{D}\gamma$) is orthogonal to the vector $\beta \times \gamma$. The additional constraint related to the uniqueness of the value of the Lagrange multiplier ν consists of satisfying the fourth of conditions (8.3).

By virtue of this condition the quantities $(\mathbf{D}\gamma, \beta)$ and $(\mathbf{A}\beta, \gamma)$ either simultaneously vanish or are simultaneously non-zero (it is assumed that $\tau \neq 0$).

The solutions can therefore be of two types (we define more accurately that we mean here by the principal axes of inertia the eigenvectors of the tensor $\mathbf{A}$).

Solutions of the first type: the vector β is a non-eigenvector of the tensor $\mathbf{A}$ and γ is a non-eigenvector of the tensor $\mathbf{D}$ (and $(\mathbf{A}\beta, \gamma) \neq 0$, $(\mathbf{D}\gamma, \beta) \neq 0$). For each of these solutions the principal axes of inertia

lie in a common position with respect to the vector β and to the vector $\overrightarrow{NC}$ (if necessary, oriented in the principal planes but not located along the principal axes) and the value of the derivative $r^3\psi^2$ is fixed (which establishes the relation between the radius of the orbit and the angular velocity).

Solutions of the second type: the vector β is an eigenvector of the tensor $\mathbf{A}$ and γ is an eigenvector of the tensor $\mathbf{D}$, while the constant τ is a certain number (and $(\mathbf{A}\beta, \gamma) = (\mathbf{D}\gamma, \beta) = 0$). For these solutions one of the principal axes of inertia is oriented along the normal to the orbital plane one of the principal directions of the tensor $\mathbf{D}$ is oriented along the vector $\overrightarrow{NC}$, and the values of the radius of the orbit and the angular velocity are arbitrary.

We will consider the first, third and fourth relations from (8.3) as a subsystem in γ. Bearing in mind the symmetry of the tensors $\mathbf{A}$ and $\mathbf{D}$ and the properties of a mixed product, we will represent the conditions for the vector $\gamma \neq 0$, which satisfies this system, to exist in the form

$$(\boldsymbol{\beta} \times (\mathbf{A}\boldsymbol{\beta} \times \boldsymbol{\beta}), \mathbf{D}\boldsymbol{\beta} + \tau\mathbf{A}\boldsymbol{\beta}) = 0 \tag{8.5}$$

In this case the solution itself can be written as (7.10). The condition that γ should satisfy the remaining of equations (8.3) has the form

$$(\boldsymbol{\beta} \times (\mathbf{A}\boldsymbol{\beta} \times \boldsymbol{\beta})), \mathbf{D}(\mathbf{A}\boldsymbol{\beta} \times \boldsymbol{\beta})) = 0 \tag{8.6}$$

Finally, in order for system (8.3) to have non-zero solutions, it is necessary and sufficient that conditions (8.5) and (8.6) should be satisfied.

Condition (8.5) defines a fourth-order cone in β space, whereas condition (8.6) defines a fifth-order cone, and both cones contain eigenvectors of the tensor $\mathbf{A}$. One solution of the first kind (β, γ, τ) (and the orthonormal basis $\{\alpha, \beta, \gamma\}$ with vector α, having the same direction as $\mathbf{A}\beta \times \beta$) corresponds to each non-zero vector β (the length of which can be assumed to be equal to unity), which belongs to this cone (8.6) and is not collinear with any one of the principal directions of inertia. On the other hand, the vectors β, collinear with at least one of the principal directions, may or may not be necessarily solutions of the second kind. The intersection of these cones with the sphere $(\beta, \beta) = 1$ is formed of points which define the possible positions of the axes in β space.

It we consider the first, second and fourth relations of (8.3) as a subsystem in β, similar discussions give the conditions for $\beta \neq 0$ to exist, which can be represented in the form

$$(\boldsymbol{\gamma} \times (\mathbf{D}\boldsymbol{\gamma} \times \boldsymbol{\gamma}), \ \mathbf{A}(\mathbf{D}\boldsymbol{\gamma} \times \boldsymbol{\gamma})) = 0, \ (\mathbf{D}\boldsymbol{\gamma} \times \boldsymbol{\gamma})^2 + \tau(\mathbf{A}\boldsymbol{\gamma} \times \boldsymbol{\gamma}, \ \mathbf{D}\boldsymbol{\gamma} \times \boldsymbol{\gamma}) = 0 \tag{8.7}$$

In this case the solution itself has the form (7.12).

An investigation of the solutions of system of equations (8.5) and (8.6) or (8.7) is fairly lengthy and requires the use of methods of algebraic geometry.

9. THE SIMPLEST RELATIVE EQUILIBRIA AND THE SUFFICIENT CONDITIONS FOR THEIR STABILITY

Suppose the tensors $\mathbf{A}$ and $\mathbf{D}$ are coaxial. Then the simplest relative equilibria exist, on which the eigenvectors of the matrices $\mathbf{A}$ and $\mathbf{D}$ coincide with the axes of an orbital system of coordinates. Suppose one of the principal axes of the matrices $\mathbf{A}$ and $\mathbf{D}$ is directed along β, while the other is directed along γ. Then, for example

$$\boldsymbol{\alpha} = (\pm 1, 0, 0), \quad \boldsymbol{\beta} = (0, \pm 1, 0), \quad \boldsymbol{\gamma} = (0, 0, \pm 1) \tag{9.1}$$

In this case

$$\nu = 0, \quad \lambda = \dot{\psi}^2 a_2, \quad \mu = -K d_3 \tag{9.2}$$

In order to obtain the sufficient conditions for the relative equilibria to be stable, it is sufficient to investigate [6, 7] the signature of the constraint of the second variation of the function W on the linear manifold

$$\delta\mathcal{F} = \{(\delta\boldsymbol{\beta}, \delta\boldsymbol{\gamma}) : (\boldsymbol{\beta}, \delta\boldsymbol{\beta}) = 0, (\boldsymbol{\gamma}, \delta\boldsymbol{\gamma}) = 0, (\boldsymbol{\beta}, \delta\boldsymbol{\gamma}) + (\boldsymbol{\gamma}, \delta\boldsymbol{\beta}) = 0\}$$

The second variation on the relative equilibria can be represented in the form

$$2\delta^2 W = \left(\frac{\partial^2 \mathcal{L}_r}{\partial \boldsymbol{\Omega}^2}\delta\boldsymbol{\Omega}, \delta\boldsymbol{\Omega}\right) + \left(\left(\frac{\partial^2 \mathcal{L}_r}{\partial \boldsymbol{\beta}^2} + \lambda\mathbf{E}\right)\delta\boldsymbol{\beta}, \delta\boldsymbol{\beta}\right) +$$

$$+\left(\left(\frac{\partial^2 \mathcal{L}_r}{\partial \boldsymbol{\gamma}^2} + \mu\mathbf{E}\right)\delta\boldsymbol{\gamma}, \delta\boldsymbol{\gamma}\right) + 2\left(\left(\frac{\partial^2 \mathcal{L}_r}{\partial \boldsymbol{\beta}\partial\boldsymbol{\gamma}} + \nu\mathbf{E}\right)\delta\boldsymbol{\gamma}, \delta\boldsymbol{\beta}\right) =$$

$$= (\mathbf{A}\delta\boldsymbol{\Omega}, \delta\boldsymbol{\Omega}) + ((\lambda\mathbf{E} - \dot{\psi}^2\mathbf{A})\delta\boldsymbol{\beta}, \delta\boldsymbol{\beta}) + ((\mu\mathbf{E} + K\mathbf{D})\delta\boldsymbol{\gamma}, \delta\boldsymbol{\gamma}) + \nu(\delta\boldsymbol{\beta}, \delta\boldsymbol{\gamma})$$

On the relative equilibria considered the linear manifold is defined by the equalities

$$\delta\beta_2 = 0, \quad \delta\gamma_3 = 0, \quad \beta_2\delta\gamma_2 + \gamma_3\delta\beta_3 = 0 \Leftrightarrow \delta\gamma_2 = \pm\delta\beta_3 = \delta \tag{9.3}$$

The constraint of the second variation on the linear manifold therefore gives

$$\begin{aligned} 2\delta^2 W|_{(9.3)} &= (\mathbf{A}\delta\boldsymbol{\Omega}, \delta\boldsymbol{\Omega}) + \dot{\psi}^2(a_2 - a_1)\delta\beta_1^2 + \dot{\psi}^2(a_2 - a_3)\delta\beta_3^2 + \\ &+ K(d_1 - d_3)\delta\gamma_1^2 + K(d_2 - d_3)\delta\gamma_2^2 = \\ &= (\mathbf{A}\delta\boldsymbol{\Omega}, \delta\boldsymbol{\Omega}) + \dot{\psi}^2(a_2 - a_1)\delta\beta_1^2 + K(d_1 - d_3)\delta\gamma_1^2 + \\ &+ [\dot{\psi}^2(a_2 - a_3) + K(d_2 - d_3)]\delta^2 \end{aligned} \tag{9.4}$$

The first term is a quadratic form, always positive-definite by virtue of the fact that the kinetic energy of the system is positive-definite. The second term is positive-definite if

$$a_2 - a_1 > 0$$

i.e. if the generalized moment of inertia about the axis normal to the orbital plane is greater than the generalized moment of inertia about the axis tangential to the orbit. The third term is positive if

$$d_1 - d_3 > 0$$

i.e. if the eigenvalue of the matrix **D** corresponding to the eigenvector directed along the tangent to the orbit is greater than the eigenvalue of the matrix **D** corresponding to the eigenvector directed along the local vertical. The latter condition of positive-definiteness has the form

$$\dot{\psi}^2(a_2 - a_3) + K(d_2 - d_3) > 0$$

Non-satisfaction of any of these conditions implies bifurcation of the solution considered. If the index of quadratic form (9.4) is odd, in other words, the degree of instability is odd, we have instability of the motion in question. If the index of this formula is even and is non-zero, it is possible for gyroscopic stabilization to exist, i.e. stability in the first approximation. This possibility will be realised if all the roots of the characteristic equation are pure imaginary.

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